

Quantum bundles and quantum interactions

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Abstract

A geometric framework for describing quantum particles on a possibly curved background is proposed. Natural constructions on certain distributional bundles (‘quantum bundles’) over the spacetime manifold yield a quantum “formalism” along any 1-dimensional timelike submanifold (a ‘detector’); in the flat, inertial case this turns out to reproduce the basic results of the usual quantum field theory, while in general it could be seen as a local, “linearized” description of the actual physics.

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Introduction

Quantisation, intended as the construction of a quantum theory by applying suitable rules to classical systems, is perhaps the most common approach to the study of the foundations of quantum physics; indeed, this philosophy has produced an immense physical and mathematical literature. There is, however, a widespread opinion that the true relation between classical and quantum theories should rather go in the opposite sense: at least in principle, classical physics should derive from quantum physics, thought to be more fundamental.

As a first step in that direction, one could try and build a stand-alone mathematical model, not derived from a quantisation procedure, which should reproduce (at least) the basic observed facts of elementary particle physics. The present article is a proposal in this sense, based on two main ingredients: *free states*, and *interaction*. A further interesting feature of the model is its freedom from the requirement of spacetime flatness.

The fundamental mathematical tool of my exploration is the geometry of *distributional bundles*, that is bundles over classical (finite-dimensional Hausdorff) manifolds whose fibres are distributional spaces. These arise naturally from a class of finite-dimensional 2-fibred bundles, which turns out to contain the most relevant physical cases. The basics of their geometry have been exposed in two previous papers [C00a, C04a] along the line of thought stemming from Frölicher's notion of smoothness [Fr82, FK88, KM97, MK98, CK95].

While I do not quantise classical fields, at this stage I do consider certain finite-dimensional geometric structures which are related to classical field theories.¹ From these one can naturally build 2-fibred bundles and, eventually, *quantum bundles*: distributional bundles whose fibres are spaces of one-particle states, and the related *Fock bundles*. It turns out that the underlying, finite-dimensional geometric structure determines a distinguished connection on a quantum bundle; this connection is related to the description of *free-particle states*.

The basic idea about quantum interactions is that they should be described by a new connection on the Fock bundle, obtained by adding an interaction morphism to the free-particle connection. This approach requires the notion of a *detector*, defined to be a timelike 1-dimensional submanifold of the spacetime manifold. Then a natural interaction morphism indeed exists in the fibres of the restricted Fock bundle. It turns out that a detector carries a quantum “formalism” which can be seen as a kind of complicated clock; in the flat, inertial case this turns out to reproduce the basic results of the usual quantum field theory,² while in general it could be seen as a local, “linearized” description of the actual physics.

The paper's plan is as follows. In the two first sections I will summarize the basic ideas about distributional bundles and quantum bundles, the latter being defined as certain bundles of generalized half-densities on classical momentum bundles; then I will introduce generalized frames for quantum bundles and the notion of a detector. In section 5 I will illustrate the construction of the quantum interaction from a general (and necessarily sketchy) point of view. In section 6 these ideas will be implemented in the simplest case, a theory of two scalar particles; in sections 7, 8, 9 and 10 I will show how to treat QED in the above said setting; in the flat inertial case one then recovers the basic known results. Here, the role of 2-fibred bundles turns out to be specially meaningful.

1 Distributional bundles

For details about the ideas reviewed in this section, see [C00a, C04a].

Let $p : \mathbf{Y} \rightarrow \mathbf{Y}$ be a real or complex classical vector bundle, namely a finite-dimensional vector bundle over the Hausdorff paracompact smooth real manifold

¹In particular gravitation, here, is a fixed background.

²The usual quantum fields can be recovered [C04b] as certain natural geometric structures of the quantum bundles, but they only play a marginal role in this approach.

\underline{Y} . Moreover assume that \underline{Y} is oriented, let $n := \dim \underline{Y}$, and consider the positive component $\mathbb{V}^* \underline{Y} := (\wedge^n T^* \underline{Y})^+ \rightarrow \underline{Y}$, called the bundle of *positive densities* on \underline{Y} .

Let $\mathcal{Y}_0 \equiv \mathcal{D}_0(\underline{Y}, \mathbb{V}^* \underline{Y} \otimes_{\underline{Y}} Y^*)$ be the vector space of all ‘test sections’, namely smooth sections $\underline{Y} \rightarrow \mathbb{V}^* \underline{Y} \otimes_{\underline{Y}} Y^*$ which have compact support. A topology on this space can be introduced by a standard procedure [Sc66]; its topological dual will be denoted as $\mathcal{Y} \equiv \mathcal{D}(\underline{Y}, Y)$ and called the space of *generalized sections*, or *distribution-sections* of the given classical bundle. Some particular cases of generalized sections are that of *r-currents* ($Y \equiv \wedge^r T^* \underline{Y}$, $r \in \mathbb{N}$) and that of *half-densities* ($Y \equiv (\mathbb{V}^* \underline{Y})^{1/2} \equiv \mathbb{V}^{-1/2} \underline{Y}$).

A curve $\alpha : \mathbb{R} \rightarrow \mathcal{Y}$ is said to be *F-smooth* if the map

$$\langle \alpha, u \rangle : \mathbb{R} \rightarrow \mathbb{C} : t \mapsto \langle \alpha(t), u \rangle$$

is smooth for every $u \in \mathcal{Y}_0$. Accordingly, a function $\phi : \mathcal{Y} \rightarrow \mathbb{C}$ is called *F-smooth* if $\phi \circ \alpha : \mathbb{R} \rightarrow \mathbb{C}$ is smooth for all F-smooth curve α . The general notion of F-smoothness, for any mapping involving distributional spaces, is introduced in terms of the standard smoothness of all maps, between finite-dimensional manifolds, which can be defined through compositions with F-smooth curves and functions. Moreover, it can be proved that a function $f : \mathbf{M} \rightarrow \mathbb{R}$, where \mathbf{M} is a classical manifold, is smooth (in the standard sense) iff the composition $f \circ c$ is a smooth function of one variable for any smooth curve $c : \mathbb{R} \rightarrow \mathbf{M}$. Thus one has a unique notion of smoothness based on smooth curves, including both classical manifolds and distributional spaces.

In the basic classical geometric setting underlying distributional bundles one considers a classical 2-fibred bundle

$$\mathbf{V} \xrightarrow{\mathbf{q}} \mathbf{E} \xrightarrow{\underline{\mathbf{q}}} \mathbf{B},$$

where $\mathbf{q} : \mathbf{V} \rightarrow \mathbf{E}$ is a vector bundle, and the fibres of the bundle $\mathbf{E} \rightarrow \mathbf{B}$ are smoothly oriented. Moreover, one assumes that $\mathbf{q} \circ \underline{\mathbf{q}} : \mathbf{V} \rightarrow \mathbf{B}$ is also a bundle, and that for any sufficiently small open subset $\mathbf{X} \subset \mathbf{B}$ there are bundle trivializations

$$(\underline{\mathbf{q}}, \underline{\mathbf{y}}) : \mathbf{E}_{\mathbf{X}} \rightarrow \mathbf{X} \times \underline{\mathbf{Y}}, \quad (\mathbf{q} \circ \underline{\mathbf{q}}, y) : \mathbf{V}_{\mathbf{X}} \rightarrow \mathbf{X} \times \mathbf{Y}$$

with the following projectability property: there exists a surjective submersion $\mathbf{p} : \mathbf{Y} \rightarrow \underline{\mathbf{Y}}$ such that the diagram

$$\begin{array}{ccc} \mathbf{V}_{\mathbf{X}} & \xrightarrow{(\mathbf{q} \circ \underline{\mathbf{q}}, y)} & \mathbf{X} \times \mathbf{Y} \\ \mathbf{q} \downarrow & & \downarrow \mathbb{I}_{\mathbf{X}} \times \mathbf{p} \\ \mathbf{E}_{\mathbf{X}} & \xrightarrow{(\underline{\mathbf{q}}, \underline{\mathbf{y}})} & \mathbf{X} \times \underline{\mathbf{Y}} \end{array}$$

commutes; this implies that $\mathbf{Y} \rightarrow \underline{\mathbf{Y}}$ is a vector bundle, which is not trivial in general.

The above conditions are easily checked to hold in many cases which are relevant for physical applications, and in particular when $V = E \times_B W$ where $W \rightarrow B$ is a vector bundle, when $V = VE$ (the vertical bundle of $E \rightarrow B$) and when V is any component of the tensor algebra of $VE \rightarrow E$.

For each $x \in B$ one considers the distributional space $\mathcal{V}_x := \mathcal{D}(E_x, V_x)$, and obtains the fibred set

$$\wp : \mathcal{V} \equiv \mathcal{D}_B(E, V) := \bigsqcup_{x \in B} \mathcal{V}_x \rightarrow B .$$

An isomorphism of vector bundles yields an isomorphism of the corresponding spaces of generalized sections; hence, a local trivialization of the underlying classical 2-fibred bundle, as above, yields a local bundle trivialization

$$(\wp, Y) : \mathcal{V}_X \rightarrow X \times \mathcal{Y} , \quad \mathcal{Y} \equiv \mathcal{D}(\underline{Y}, Y)$$

of $\mathcal{V} \rightarrow B$. Moreover, a smooth atlas of 2-bundle trivializations determines a linear F-smooth bundle atlas on $\mathcal{V} \rightarrow B$, which is said to be an *F-smooth distributional bundle*. In general, the F-smoothness of any map from or to \mathcal{V} is equivalent to the F-smoothness of its local trivialized expression.

One defines the *tangent space* of any F-smooth space through equivalence classes of F-smooth curves; tangent prolongations of any F-smooth mappings can also be shown to exist. Thus one gets, in particular, the tangent space $T\mathcal{V}$, which has local trivializations as $TX \times T\mathcal{Y}$, its *vertical subspace* and the *first jet bundle* $J\mathcal{V} \rightarrow \mathcal{V}$. A *connection* is defined to be an F-smooth section $\mathfrak{G} : \mathcal{V} \rightarrow J\mathcal{V}$.

With some care, many of the usual chart expressions of finite-dimensional differential geometry can be extended to the distributional case. In particular, let $\sigma : B \rightarrow \mathcal{V}$ be an F-smooth section and $\sigma^\vee := Y \circ \sigma : B \rightarrow \mathcal{Y}$ its ‘chart expression’. Then its covariant derivative has the chart expression

$$(\nabla \sigma)^\vee = \dot{x}^a (\partial_a \sigma^\vee - \mathfrak{G}_{\vee a} \sigma^\vee) ,$$

where (x^a) is a chart on $X \subset B$ and $\mathfrak{G}_{\vee a} : X \rightarrow \text{End}(\mathcal{Y})$, $a = 1, \dots, \dim B$.

The notions of *curvature* and of *adjoint connection* can also be introduced. Furthermore, it can be shown that any projectable connection on the underlying classical 2-bundle determines a distributional connection; however, not all distributional connections arise from classical ones.

2 Quantum bundles

Let \mathbb{L} be the semi-vector space of *length units* (see [CJM95, C00b] for a review of unit spaces) and (M, g) a spacetime. The spacetime metric g has ‘conformal weight’ $\mathbb{L}^2 \cong \mathbb{L} \otimes \mathbb{L}$, namely it is a bilinear map $TM \times_M TM \rightarrow \mathbb{L}^2$, while its inverse $g^\#$ has conformal weight $\mathbb{L}^{-2} \cong \mathbb{L}^* \otimes \mathbb{L}^*$.

For $m \in \mathbb{L}^{-1} \cong \mathbb{L}^*$ let $\mathbf{P}_m \cong \mathbf{K}_m^+ \subset \mathbf{T}^*\mathbf{M}$ be the subbundle over \mathbf{M} of all future-pointing $p \in \mathbf{T}^*\mathbf{M}$ such that³ $g^\#(p, p) = m^2$. Then \mathbf{P}_m is the classical *phase bundle* for a particle of mass m ; the case $m = 0$ can be also considered. Furthermore, consider the 2-fibred bundle

$$[(\wedge^3 \mathbf{T}^*\mathbf{P}_m)^+]^{1/2} \equiv \mathbb{V}^{-1/2} \mathbf{P}_m \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$$

whose upper fibres are the spaces of half-densities on the fibres of $\mathbf{P}_m \rightarrow \mathbf{M}$. There is a distinguished section

$$\sqrt{\omega_m} : \mathbf{P}_m \rightarrow \mathbb{L}^{-1} \otimes \mathbb{V}^{-1/2} \mathbf{P}_m ;$$

here, ω_m is the *Leray form* of the hyperboloids (the fibres of $\mathbf{P}_m \rightarrow \mathbf{M}$), usually indicated as $\delta(g^\diamond - m^2)$ where g^\diamond is the contravariant quadratic form associated with the metric. If $(\mathbf{p}_\lambda) = (\mathbf{p}_0, \mathbf{p}_i)$ are \mathbb{L}^{-1} -scaled orthonormal coordinates on the fibres of $\mathbf{T}^*\mathbf{M} \rightarrow \mathbf{M}$, then one finds the coordinate expression

$$\sqrt{\omega_m} = \frac{\sqrt{d^3 \mathbf{p}_\perp}}{\sqrt{2 E_m}} ,$$

where $E_m := \sqrt{m^2 + |\mathbf{p}_\perp|^2} = \sqrt{m^2 + \delta^{ij} \mathbf{p}_i \mathbf{p}_j}$ is indicated simply as \mathbf{p}_0 if no confusion arises. Note that $d^3 \mathbf{p}_\perp \equiv d\mathbf{p}_1 \wedge d\mathbf{p}_2 \wedge d\mathbf{p}_3$ is the “spatial” (scaled) volume form determined by the “observer” associated with the coordinates, and can be seen as a volume form on the fibres of \mathbf{P}_m via orthogonal projection.

It can be seen [C04a] that the spacetime connection Γ determines a connection Γ_m of $\mathbf{P}_m \rightarrow \mathbf{M}$, as well as a linear connection of the 2-fibred bundle $\mathbb{V}^*\mathbf{P}_m \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$ which is projectable on Γ_m ; on turn this determines a linear projectable connection $\hat{\Gamma}_m$ of $\mathbb{V}^{-1/2} \mathbf{P}_m \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$, with the coordinate expression

$$(\Gamma_m)_{ai} = -\Gamma_{ai}^0 \mathbf{p}_0 - \Gamma_{ai}^j \mathbf{p}_j , \quad (\hat{\Gamma}_m)_a = -\frac{\Gamma_{ai}^0 g^{ij} \mathbf{p}_j}{\mathbf{p}_0} + \frac{1}{2} \Gamma_{ai}^i$$

(here a is an index for coordinates on \mathbf{M} and the spacelike coordinates (\mathbf{p}_j) play the role of fibre coordinates on $\mathbf{P}_m \rightarrow \mathbf{M}$).

Next consider the distributional bundle

$$\mathcal{P}_m := \mathcal{P}_M(\mathbf{P}_m) \equiv \mathcal{D}_M(\mathbf{P}_m, \mathbb{C} \otimes \mathbb{V}^{-1/2} \mathbf{P}_m) \rightarrow \mathbf{M} ,$$

whose fibre over each $x \in \mathbf{M}$ is the vector space of all (complex-valued) generalized half-densities on $(\mathbf{P}_m)_x$. The connection $\hat{\Gamma}_m$ determines a smooth (in Frölicher’s sense) connection $\mathcal{P}_m \rightarrow \mathbf{M}$ which can be characterized in various ways [C04a], the most simple being the following: let $c : \mathbb{R} \rightarrow \mathbf{M}$ be any local curve and $p : \mathbf{M} \rightarrow \mathbf{P}_m$ a local section which is parallelly transported along c ; then the local section

$$\delta_p \otimes (\omega_m)^{-1/2} : \mathbf{M} \rightarrow \mathbb{L} \otimes \mathcal{P}_m : x \mapsto \delta_{p(x)} \otimes (\omega_m)^{-1/2} , \quad x \in \mathbf{M}$$

³Throughout this paper, the signature of the metric is $(1, 3)$; moreover $\hbar = c = 1$, so that \mathbb{L} is the unique unit space involved.

is parallelly transported along c , where $\delta_{p(x)}$ denotes the Dirac density on $(\mathbf{P}_m)_x$ whose support is the point $p(x)$.

Let now $\mathbf{V} \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$ be a (real or complex) 2-fibred vector bundle, and consider the distributional bundle

$$\mathbf{V}^1 := \mathcal{D}_M(\mathbf{P}_m, \mathbf{V}) \equiv \mathcal{D}_M(\mathbf{P}_m, \mathbb{V}^{-1/2} \mathbf{P}_m \otimes \mathbf{V}) \rightarrow \mathbf{M} ,$$

whose fibre over each $x \in \mathbf{M}$ is the vector space of all \mathbf{V} -valued generalized half-densities on $(\mathbf{P}_m)_x$. In practice, this \mathbf{V} will be related to the bundle whose sections are the fields of the classical theory which, in the usual approach, correspond to the quantum theory under consideration. One could think that it suffices to deal with a “semi-trivial” 2-fibred bundle $\mathbf{P}_m \times_M \mathbf{V}$ where $\mathbf{V} \rightarrow \mathbf{M}$ is a vector bundle, however it will be seen (§10) that the general setting is actually needed.

Remark. If a Hermitian metric on the fibres of \mathbf{V} is given, then one can define a *Hilbert bundle* $\mathcal{H} \rightarrow \mathbf{M}$, and has inclusions $\mathbf{V}_\circ^1 \subset \mathcal{H} \subset \mathbf{V}^1$ (where $\mathbf{V}_\circ^1 \rightarrow \mathbf{M}$ is the subbundle whose fibres are constituted by test sections); namely one has a bundle of ‘rigged Hilbert spaces’ [BLT75].

A *Fock bundle* can be constructed as

$$\mathbf{V} := \bigoplus_{j=0}^{\infty} \mathbf{V}^j ,$$

where

$$\text{either } \mathbf{V}^j := \wedge^j \mathbf{V}^1 \quad \text{or} \quad \mathbf{V}^j := \vee^j \mathbf{V}^1$$

(antisymmetrized and symmetrized tensor products).

If a connection γ of $\mathbf{V} \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$ linear projectable over Γ_m is given (which is the case in most physical situations), then one also gets a connection of $\mathbf{V}^1 \rightarrow \mathbf{M}$; this can be naturally extended to a connection on $\mathbf{V} \rightarrow \mathbf{M}$, which will be called the *free particle connection*. For any local section $\sigma : \mathbf{M} \rightarrow \mathbf{V}^1$ one has the coordinate expression

$$\mathfrak{C}_a^A(\sigma) = \gamma_a^A{}_B \sigma^B - (\Gamma_m)_{ai} \partial^i \sigma^A .$$

3 Generalized frames

For each $p \in (\mathbf{P}_m)_x$, $x \in \mathbf{M}$, let $\delta[p]$ denote the Dirac generalized density on the fibre $(\mathbf{P}_m)_x$ with support $\{p\}$; namely, if $f : (\mathbf{P}_m)_x \rightarrow \mathbb{C}$ is a test function then one has $\langle \delta[p], f \rangle = f(p)$. It can be written as⁴

$$\delta[p] = \check{\delta}[p] d^3 \mathbf{p}_\perp$$

⁴While p denotes an element of \mathbf{P}_m , the sans-serif symbol \mathbf{p} is used for the fibre coordinates.

where $\check{\delta}[p]$ is the \mathbb{L}^3 -scaled *generalized function*, usually denoted as $\check{\delta}[p](q) \equiv \delta(q-p)$, acting on test densities $\phi = \check{\phi} d^3 p_\perp$ as

$$\langle \check{\delta}[p], \phi \rangle = \langle \delta[p], \check{\phi} \rangle = \check{\phi}(p) .$$

Actually any generalized density can be expressed in this way as a generalized function times a given volume form; moreover, note that the spacelike volume form $d^3 p_\perp$, as well as the induced volume form on the fibres of $\mathbf{P}_m \rightarrow \mathbf{M}$ denoted in the same way, only depends on the choice of an ‘observer’ (i.e. a timelike future-pointing unit vector field) and not on the particular frame of $T^*\mathbf{M}$ adapted to it.

Let now $l \in \mathbb{L}$ be an arbitrarily fixed length unit, and consider the unscaled generalized half-density

$$\mathbf{B}_p := l^{-3/2} \check{\delta}[p] \sqrt{d^3 p_\perp} = \frac{1}{\sqrt{2 l^3 p_0}} \delta[p] \otimes \omega_m^{-1/2} ,$$

acting on test half-densities $\theta = \check{\theta} \sqrt{d^3 p_\perp}$ as

$$\langle \mathbf{B}_p, \theta \rangle = l^{-3/2} \langle \delta[p], \check{\theta} \rangle = l^{-3/2} \check{\theta}(p) ;$$

for each $x \in \mathbf{M}$ the set $\{\mathbf{B}_p\}$, $p \in (\mathbf{P}_m)_x$, can be seen as a *generalized frame* of the distributional bundle \mathcal{P}_m at x . Let moreover $\{\mathbf{b}_A\}$ be a frame of the classical vector bundle $\mathbf{V} \rightarrow \mathbf{P}_m$, $A = 1, \dots, n$; then

$$\{\mathbf{B}_{pA}\} := \{\mathbf{B}_p \otimes \mathbf{b}_A\}$$

is a generalized frame of $\mathcal{V}^1 \rightarrow \mathbf{M}$. In fact, any $\psi = \psi^A \mathbf{b}_A \in \mathcal{V}^1$ can be written as $\int \psi^A(\mathbf{p}) \mathbf{B}_{pA}$, which is to be intended in the generalized sense

$$\langle \psi, \theta \rangle = \int \check{\psi}^A(\mathbf{p}) \check{\theta}_A(\mathbf{p}) d^3 p_\perp$$

where $\theta \in \mathcal{V}_0^1$ is a test half density in the same fibre as ψ .

Let \mathcal{A} be a set (index set); a *generalized multi-index* is defined to be a map

$$I : \mathcal{A} \rightarrow \{0\} \cup \mathbb{N}$$

vanishing outside some finite subset $\mathcal{A}_I \subset \mathcal{A}$; it can be represented through its graphic

$$\{(\alpha_1, I_1), (\alpha_2, I_2), \dots, (\alpha_r, I_r)\} , \quad \mathcal{A}_I = \{\alpha_1, \dots, \alpha_r\}$$

for any (arbitrary and inessential) ordering of \mathcal{A}_I . Now one extends the generalized frame $\{\mathbf{B}_{pA}\}$ to a generalized frame of the Fock bundle $\mathcal{V} \rightarrow \mathbf{M}$ by letting $\mathcal{A}_x = (\mathbf{P}_m)_x \times \{1, \dots, n\}$ for each $x \in \mathbf{M}$, and setting

$$\mathbf{B}_I := \frac{(\mathbf{B}_{\alpha_1})^{I_1} \vee (\mathbf{B}_{\alpha_2})^{I_2} \vee \dots (\mathbf{B}_{\alpha_r})^{I_r}}{\sqrt{I_1! I_2! \dots I_r!}} \quad (\text{bosons}),$$

$$\mathbf{B}_I := \mathbf{B}_{\alpha_1} \wedge \mathbf{B}_{\alpha_2} \wedge \dots \wedge \mathbf{B}_{\alpha_r} \quad (\text{fermions}),$$

where

$$(\mathbf{B}_\alpha)^k := \underbrace{\mathbf{B}_\alpha \vee \cdots \vee \mathbf{B}_\alpha}_{k \text{ times}} .$$

In a more detailed way one writes $\alpha_i = (p_i, A_i)$ and

$$\mathbf{B}_I := \frac{(\mathbf{B}_{p_1 A_1})^{I_1} \vee (\mathbf{B}_{p_2 A_2})^{I_2} \vee \cdots (\mathbf{B}_{p_r A_r})^{I_r}}{\sqrt{I_1! I_2! \cdots I_r!}} \quad (\text{bosons}),$$

$$\mathbf{B}_I := \mathbf{B}_{p_1 A_1} \wedge \mathbf{B}_{p_2 A_2} \wedge \cdots \wedge \mathbf{B}_{p_r A_r} \quad (\text{fermions}).$$

If one has a Hermitian structure in the fibres of $\mathbf{V} \rightarrow \mathbf{P}_m$ and $\{\mathbf{b}_A\}$ is an orthonormal classical frame, then one gets an ‘orthonormality’ relation $\langle \mathbf{B}_I, \mathbf{B}_J \rangle = \delta_{IJ}$, to be interpreted in a generalized (i.e. distributional) sense.

4 Detectors

By a ‘detector’ I mean a 1-dimensional time-like submanifold $\mathbf{T} \subset \mathbf{M}$. Locally this determines, via the exponentiation map, a time+space splitting, which in a sense relates the momentum-space based approach presented here to a position-space approach, though the relation is precise only if the induced splitting is global.

Consider restrictions of the quantum bundles previously introduced to bundles over \mathbf{T} , so write

$$\mathcal{P}_m \rightarrow \mathbf{T}, \quad \mathcal{V}^1 \rightarrow \mathbf{T}, \quad \mathcal{V} \rightarrow \mathbf{T},$$

and the like. Clearly, the free particle connection determines connections of these bundles; it actually turns out that one gets (possibly local) splittings of them. So one writes, for example

$$\mathcal{V} \cong \mathbf{T} \times \mathcal{V}_{t_0}$$

where $t_0 \in \mathbf{T}$ is some arbitrarily chosen point. Note that the free particle connection, by construction, preserves “particle number”, namely is reducible to a connection of each of the subbundles \mathcal{V}^j , $j \in \{0\} \cup \mathbb{N}$.

The unit future-pointing vector field $\Theta_0 : \mathbf{T} \rightarrow \mathbb{L} \otimes (\mathbf{T}^* \mathbf{M})_{\mathbf{T}}$ tangent to \mathbf{T} determines an orthogonal splitting $(\mathbf{T}^* \mathbf{M})_{\mathbf{T}} = \mathbf{T}^* \mathbf{T} \times_{\mathbf{T}} (\mathbf{T}^* \mathbf{M})_{\mathbf{T}}^\perp$, and a diffeomorphism $(\mathbf{P}_m)_{\mathbf{T}} \leftrightarrow (\mathbf{T}^* \mathbf{M})_{\mathbf{T}}^\perp$; the ‘spacelike’ volume form on the fibres of $\mathbb{L} \otimes (\mathbf{T}^* \mathbf{M})_{\mathbf{T}}^\perp$ then yields a scaled volume form on the fibres of $(\mathbf{P}_m)_{\mathbf{T}}$; with the choice of a length unit one obtains a generalized frame $\{\mathbf{B}_{pA}\}$ of $\mathcal{V}^1 \rightarrow \mathbf{T}$; in practice, this is defined in the same way as the generalized frame of $\mathcal{V}^1 \rightarrow \mathbf{M}$ introduced in §3, where now the orthonormal coordinates $(\mathbf{p}_\lambda) \equiv (\mathbf{p}_0, \mathbf{p}_i)$ are *adapted* to the above said splitting.

Let now $p : \mathbf{T} \rightarrow (\mathbf{P}_m)_{\mathbf{T}}$ be a covariantly constant section and $(\mathbf{b}_A(p))$ a frame of $\mathbf{V} \rightarrow \mathbf{P}_m$ covariantly constant over p . If $\mathbf{T} \subset \mathbf{M}$ is a geodesic submanifold, then \mathbf{B}_{pA} is covariantly constant along \mathbf{T} relatively to the free-particle connection. Thus the generalized orthonormal set $\{\mathbf{B}_{pA}\}$ indexed by covariantly constant sections $p : \mathbf{T} \rightarrow (\mathbf{P}_m)_{\mathbf{T}}$ and by the classical index A is constant in the same sense. If \mathbf{T} is not

geodesic then one can either construct the generalized frame at some chosen $t_0 \in \mathbf{T}$ and then parallelly propagate it along \mathbf{T} , or modify the definition of the free-particle connection of $\mathcal{V} \rightarrow \mathbf{T}$ by relating it to *Fermi transport* rather than parallel transport along \mathbf{T} . From the physical point of view one may expect different interpretations of these two settings, which however give rise essentially to the same formalism.

5 Quantum interaction

The general idea of quantum interaction is the following. Consider a Fock bundle $\mathcal{V} = \mathcal{V}' \otimes_{\mathbf{T}} \mathcal{V}'' \otimes_{\mathbf{T}} \mathcal{V}''' \otimes_{\mathbf{T}} \dots$ (each factor being, on turn, a Fock bundle accounting for a given particle type) endowed with a free-particle connection \mathfrak{G} . Suppose that there exists a distinguished section $\mathfrak{H} : \mathbf{T} \rightarrow \mathbb{L}^{-1} \otimes \text{End}(\mathcal{V})$; by considering the unit future-oriented section $dt : \mathbf{T} \rightarrow \mathbb{L} \otimes T^*\mathbf{T}$ determined via the spacetime metric, one can introduce a new connection $\mathfrak{G} - i\mathfrak{H} dt$ (possibly mixing particle numbers and types). A section $\mathbf{T} \rightarrow \mathcal{V}$ which is constant relatively to $\mathfrak{G} - i\mathfrak{H} dt$ describes the evolution of a particle system, or rather the evolution of a quantum *clock* (in a broad sense) of the detector. This evolution can be compared to that determined by \mathfrak{G} alone, namely it can be read in a fixed Fock *space* $\mathcal{Y} \equiv \mathcal{V}_{t_0}$, $t_0 \in \mathbf{T}$, such that $\mathcal{V} \cong \mathbf{T} \times \mathcal{Y}$ is the splitting determined by \mathfrak{G} .⁵

Now the evolution operator $\mathcal{U}_{t_0} : \mathbf{T} \rightarrow \text{End}(\mathcal{Y})$ can be written as the formal series

$$\mathcal{U}_{t_0}(t) = \mathbb{1} + \sum_{N=1}^{\infty} \frac{(-i)^N}{N!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{N-1}} dt_N |\mathfrak{H}(t_1) \mathfrak{H}(t_2) \dots \mathfrak{H}(t_N)| ,$$

where $|\cdot|$ denotes the *time-ordered product*; the *scattering operator* is defined to be $\mathcal{S} := \mathcal{U}_{-\infty}(+\infty) \in \text{End}(\mathcal{Y})$. However, besides any convergence questions, the basic problem is the existence of \mathfrak{H} ; actually I'm going to show that there is a natural way of introducing it, and a way which is consistent with the results of the standard theory, but only as a morphism $\mathcal{V}_o \rightarrow \mathcal{V}$ (where $\mathcal{V}_o \subset \mathcal{V}$ is the subbundle of test elements). This implies that many *single terms* of the above series are not defined. Nevertheless, parts of it give considerable information which turns out to be physically true, at least in the standard, flat spacetime situation. Furthermore, in some way \mathcal{S} turns out to be well-defined in renormalizable theories.

In the rest of this section I will expose the basic ideas for the construction of \mathfrak{H} .

For each $m \in \{0\} \cup \mathbb{L}^{-1}$ the spacetime geometry yields \mathbb{L}^{-3} -scaled volume forms $\eta_m = d^3\mathbf{p}_\perp$ on the fibres of $(T^*\mathbf{M})^\perp \rightarrow \mathbf{T}$, giving rise to equally scaled volume forms, denoted by the same symbols, on the fibres of $\mathbf{P}_m \rightarrow \mathbf{T}$.

Now for $m', m'', m''' \in \{0\} \cup \mathbb{L}^{-1}$ consider the bundle ‘of three momenta’

$$\mathbf{P}_\Delta := \mathbf{P}_{m'} \times_{\mathbf{M}} \mathbf{P}_{m''} \times_{\mathbf{M}} \mathbf{P}_{m'''} \rightarrow \mathbf{M} ,$$

⁵Related ideas, describing the evolution of a quantum system in terms of a connection on a functional bundle in a Galileian setting, have been introduced in [JM02, CJM95].

and the section of *scaled* densities

$$\delta_\Delta : \mathbf{T} \rightarrow \mathbb{L}^{-6} \otimes \mathcal{D}(\mathbf{P}_\Delta) = \mathbb{L}^{-6} \otimes \mathcal{D}(\mathbf{P}_{m'}) \otimes_{\mathbf{T}} \mathcal{D}(\mathbf{P}_{m''}) \otimes_{\mathbf{T}} \mathcal{D}(\mathbf{P}_{m'''}),$$

given by

$$\begin{aligned} \langle \delta_\Delta, f \rangle &:= \iint f(\mathbf{p}'_\perp, \mathbf{p}''_\perp, -\mathbf{p}'_\perp - \mathbf{p}''_\perp) \eta_{m'} \wedge \eta_{m''} = \iint f(\mathbf{p}'_\perp, \mathbf{p}''_\perp, -\mathbf{p}'_\perp - \mathbf{p}''_\perp) d^3 \mathbf{p}' d^3 \mathbf{p}'' \\ &= \iint f(\mathbf{p}'_\perp, \mathbf{p}''_\perp, \mathbf{p}'''_\perp) \delta(\mathbf{p}'_\perp + \mathbf{p}''_\perp + \mathbf{p}'''_\perp) d^3 \mathbf{p}' d^3 \mathbf{p}'' d^3 \mathbf{p}''', \end{aligned}$$

where $f \in \mathcal{D}_0(\mathbf{P}_\Delta)$. It can be written in the form

$$\delta_\Delta = \check{\delta} \eta_{m'} \otimes \eta_{m''} \otimes \eta_{m'''} = \check{\delta}_\Delta d^3 \mathbf{p}' \otimes d^3 \mathbf{p}'' \otimes d^3 \mathbf{p}''',$$

where $\check{\delta}_\Delta \equiv \delta(\mathbf{p}'_\perp + \mathbf{p}''_\perp + \mathbf{p}'''_\perp)$ is an \mathbb{L}^3 -valued generalized function.

Now one introduces the true (unscaled) generalized half-density

$$\underline{\Delta} := \check{\delta}_\Delta \sqrt{\omega_{m'}} \otimes \sqrt{\omega_{m''}} \otimes \sqrt{\omega_{m'''}} : \mathbf{T} \rightarrow \mathcal{P}(\mathbf{P}_\Delta) = \mathcal{P}(\mathbf{P}_{m'}) \otimes_{\mathbf{T}} \mathcal{P}(\mathbf{P}_{m''}) \otimes_{\mathbf{T}} \mathcal{P}(\mathbf{P}_{m'''}),$$

which has the coordinate expression

$$\underline{\Delta} = \frac{\delta(\mathbf{p}'_\perp + \mathbf{p}''_\perp + \mathbf{p}'''_\perp)}{\sqrt{8 \mathbf{p}'_0 \mathbf{p}''_0 \mathbf{p}'''_0}} \sqrt{d^3 \mathbf{p}'} \otimes \sqrt{d^3 \mathbf{p}''} \otimes \sqrt{d^3 \mathbf{p}''' }.$$

The fact that $\underline{\Delta}$ is unscaled, independently of the choice of a length unit, will turn out to be essential for its role in the quantum interaction; here it will describe the interaction of three particles, but clearly it can be readily generalized for describing the interaction of any number of particles. The different particle types are characterized by different complex 2-fibred bundles $\mathbf{V}' \rightarrow \mathbf{P}_{m'} \rightarrow \mathbf{M}$ and the like, and one must have a ‘classical interaction Lagrangian’ that is a scalar-valued 3-linear contraction among the fibres; this is a section

$$\ell_{\text{int}} : \mathbf{P}_\Delta \rightarrow \mathbf{V}'^\star \otimes_{\mathbf{P}_\Delta} \mathbf{V}''^\star \otimes_{\mathbf{P}_\Delta} \mathbf{V}'''^\star,$$

which can be seen as 3-linear fibred contraction. The structure of these bundles must allow for ‘index raising and lowering’, thus yielding a number of objects related to ℓ_{int} and distinguished by various combinations of index types. Of course these arise in the easiest way when one has fibred Hermitian structures of the considered bundles (the fundamental case of electrodynamics, however, will be seen [§10] to be somewhat more involved). In particular, $\ell_{\text{int}}^\dagger : \mathbf{P}_\Delta \rightarrow \mathbf{V}' \otimes_{\mathbf{P}_\Delta} \mathbf{V}'' \otimes_{\mathbf{P}_\Delta} \mathbf{V}'''$.

Now one gets a section

$$\Lambda := \underline{\Delta} \otimes \ell_{\text{int}}^\dagger : \mathbf{T} \rightarrow \mathcal{P}_{\mathbf{T}}(\mathbf{P}_\Delta, \mathbf{V}' \otimes_{\mathbf{P}_\Delta} \mathbf{V}'' \otimes_{\mathbf{P}_\Delta} \mathbf{V}''') \equiv \mathcal{V}^{\prime 1} \otimes_{\mathbf{T}} \mathcal{V}^{\prime\prime 1} \otimes_{\mathbf{T}} \mathcal{V}^{\prime\prime\prime 1},$$

where $\mathcal{V}'^1 := \mathcal{P}_T(P_\Delta, V')$ and the like. The essential idea of the quantum interaction is then the following: make Λ act in the fibres of the Fock bundle $\mathcal{V} \equiv \mathcal{V}' \otimes_T \mathcal{V}'' \otimes_T \mathcal{V}''' \rightarrow T$ by using each one of its tensor factors either as ‘absorption’ (contraction) or as ‘creation’ (tensor product). However, a fundamental issue is immediately apparent (and will be furtherly discussed later on): in general, this action is only well-defined on the subbundle $\mathcal{V}_0 \subset \mathcal{V}$ of test elements, so actually it gives rise to a morphism $\mathcal{V}_0 \rightarrow \mathcal{V}$ whose extendibility will have to be carefully examined.

The various ‘index types’ of ℓ_{int} correspond to the various actions performed by the corresponding tensor factors: a covariant index determines a particle absorption, a contravariant index determines a particle creation. Furthermore one considers different types of $\underline{\Lambda}$, each one to be coupled to a corresponding type of ℓ_{int} and obtained by changing the *sign* of the momenta in the δ generalized function. So, for example, the type of ℓ_{int} which is a section $P_\Delta \rightarrow V'^\star \otimes_{P_\Delta} V'' \otimes_{P_\Delta} V'''$ (the first factor is an absorption factor, the second and third are creation factors) is tensorialized by the ‘version’ of $\underline{\Lambda}$ which has $\delta(-p'_\perp + p''_\perp + p'''_\perp)$ in its coordinate expression. In practice, I find it convenient using a ‘generalized index’ notation in which generalized indices are either high or low, and repeated momentum indices are to interpreted as integration indices (just as repeated ordinary indices are interpreted as ordinary summation indices). So, for $f = \check{f} \sqrt{d^3 p'} \otimes \sqrt{d^3 p''} \otimes \sqrt{d^3 p'''} \in \mathcal{V}_0$, I’ll write

$$\begin{aligned} f &= f_{p'p''p'''} B^{p'} \otimes B^{p''} \otimes B^{p'''} , & f_{p'p''p'''} &= l^{-9/2} \check{f}(p', p'', p''') , \\ \underline{\Lambda} &= \underline{\Lambda}^{p'p''p'''} B_{p'} \otimes B_{p''} \otimes B_{p'''} , & \underline{\Lambda}^{p'p''p'''} &= \frac{\delta(p'_\perp + p''_\perp + p'''_\perp)}{\sqrt{8 l^9 p'_0 p''_0 p'''_0}} \end{aligned}$$

(where $B^p \equiv B_p$), to be intended as

$$\langle \underline{\Lambda}, f \rangle = \underline{\Lambda}^{p'p''p'''} f_{p'p''p'''} = \int \frac{\delta(p'_\perp + p''_\perp + p'''_\perp)}{\sqrt{8 p'_0 p''_0 p'''_0}} \check{f}(p', p'', p''') d^3 p' d^3 p'' d^3 p'''$$

(remark: the generalized index summation is performed via the unscaled volume form $l^3 d^3 p_\perp$).

Analogously, the various index types of $\underline{\Lambda}$ (there are 8 of them) can be written as

$$\underline{\Lambda}_{p'p''p'''} B^{p'} \otimes B_{p''} \otimes B_{p'''} , \quad \underline{\Lambda}_{p'p''p'''} B^{p'} \otimes B^{p''} \otimes B_{p'''} , \quad \dots \quad \text{etcetera},$$

where

$$\underline{\Lambda}_{p'p''p'''} = \frac{\delta(-p'_\perp + p''_\perp + p'''_\perp)}{\sqrt{8 p'_0 p''_0 p'''_0}} , \quad \underline{\Lambda}_{p'p''p'''} = \frac{\delta(-p'_\perp - p''_\perp + p'''_\perp)}{\sqrt{8 p'_0 p''_0 p'''_0}} ,$$

and so on. Correspondingly, the various types of Λ can be written as

$$\begin{aligned} \Lambda^{p'A', p''A'', p'''A'''} B_{p'A'} \otimes B_{p''A''} \otimes B_{p'''A'''} , & \quad \Lambda^{p'A', p''A'', p'''A'''} = \Lambda^{p'p''p'''} (\ell_{\text{int}})^{A'A''A'''} , \\ \Lambda_{p'A', p''A'', p'''A'''} B^{p'A'} \otimes B_{p''A''} \otimes B_{p'''A'''} , & \quad \Lambda_{p'A', p''A'', p'''A'''} = \Lambda_{p'p''p'''} (\ell_{\text{int}})_{A'A''A'''} , \\ \dots\dots\dots & \quad \dots\dots\dots \\ \Lambda_{p'A', p''A'', p'''A'''} B^{p'A'} \otimes B^{p''A''} \otimes B^{p'''A'''} , & \quad \Lambda_{p'A', p''A'', p'''A'''} = \Lambda_{p'p''p'''} (\ell_{\text{int}})_{A'A''A'''} . \end{aligned}$$

The morphism $\mathfrak{H} : \mathcal{V}_0 \rightarrow \mathbb{L}^{-1} \otimes \mathcal{V}$ is essentially a sum whose terms are the various types of Λ , with a further ingredient: each term also has a factor

$$\lambda e^{-i(\pm p'_0 \pm p''_0 \pm p'''_0)t},$$

where $\lambda \in \mathbb{L}^{-1}$ is a constant; the signs in the exponential match those of the corresponding spatial momenta. Then $-i\mathfrak{H} \otimes dt : \mathbf{T} \rightarrow \mathcal{V}_0 \otimes \mathcal{V} \otimes T^*\mathbf{T}$ is the interaction term which modifies the free-particle connection.

The reader will note that, according to the setting above sketched, the elements B_{pA} and B^{pA} in the various generalized frames can be thought of, essentially, as the usual creation and absorption operators. Moreover one could obtain further types of Λ by exchanging tensor factors; this only make a difference if particles of the same type are involved, and is settled by considering only those terms in which the creation operators stand on the right.

6 Scalar particles

Let's see how, in practice, the somewhat sketchy ideas exposed in §5 can be implemented in the simplest case. Many of the arguments used in this section are more or less standard, the point is to show how they arise from a not-so-standard approach.

Consider a model of two types of scalar particles, one of mass m and one massless, with one-particle state bundles $\mathcal{V}^1 \equiv \mathcal{P}_m$ and $\mathcal{V}''^1 \equiv \mathcal{P}_0$ and generalized frames $\{A_p\}$, $p \in \mathbf{P}_m$, and $\{B_k\}$, $k \in \mathbf{P}_0$. The classical interaction is assumed to be just a constant $\ell \in \mathbb{L}^{-1}$, and it incorporates the λ introduced above.

At first-order, the formal series expression of the scattering operator is $\mathcal{S} = \mathbb{I} + \mathcal{S}_1$ with $\mathcal{S}_1 = -i \int_{-\infty}^{+\infty} \mathfrak{H}(t) dt$. In terms of generalized index notation, one says that \mathcal{S}_1 has a ‘matrix element’

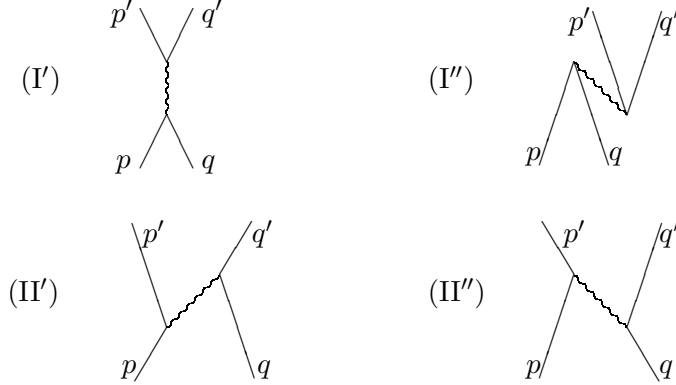
$$(\mathcal{S}_1)_{pq}^k = \frac{-i\ell \delta(-p_\perp + k_\perp - q_\perp)}{\sqrt{8l^9 k_0 p_0 q_0}} \int_{-\infty}^{+\infty} e^{i(k_0 - p_0 - q_0)t} dt = \frac{-2\pi i \ell \delta(k - p - q)}{\sqrt{8l^9 k_0 p_0 q_0}}.$$

There are eight matrix elements of this kind, each one describing *one-point interaction* and labelled by an elementary Feynman graph (time running *upwards*):

$$\begin{array}{ll} \text{Diagram 1: } \begin{array}{c} \diagup \\ \vdots \end{array} & (\mathcal{S}_1)_{pqk} = \frac{-2\pi i \ell \delta(-p - k - q)}{\sqrt{8l^9 p_0 q_0 k_0}}, & \text{Diagram 2: } \begin{array}{c} \diagdown \\ \vdots \end{array} & (\mathcal{S}_1)_{qk}^p = \frac{-2\pi i \ell \delta(p - q - k)}{\sqrt{8l^9 p_0 q_0 k_0}}, \\ \text{Diagram 3: } \begin{array}{c} \vdots \\ \diagdown \end{array} & (\mathcal{S}_1)_{pq}^k = \frac{-2\pi i \ell \delta(k - p - q)}{\sqrt{8l^9 k_0 p_0 q_0}}, & \text{Diagram 4: } \begin{array}{c} \diagup \\ \vdots \end{array} & (\mathcal{S}_1)_{pk}^q = \frac{-2\pi i \ell \delta(q - p - k)}{\sqrt{8l^9 k_0 p_0 q_0}}, \\ \text{Diagram 5: } \begin{array}{c} \vdots \\ \diagup \end{array} & (\mathcal{S}_1)_{qk}^p = \frac{-2\pi i \ell \delta(k + p - q)}{\sqrt{8l^9 k_0 p_0 q_0}}, & \text{Diagram 6: } \begin{array}{c} \diagdown \\ \vdots \end{array} & (\mathcal{S}_1)_{pk}^q = \frac{-2\pi i \ell \delta(p + q - k)}{\sqrt{8l^9 k_0 p_0 q_0}}, \end{array}$$

$$\text{---}\diagup\diagdown\text{---} \quad (\mathcal{S}_1)_{\mathbf{p}}^{\mathbf{kq}} = \frac{-2\pi i \ell \delta(\mathbf{k} + \mathbf{q} - \mathbf{p})}{\sqrt{8 l^9 k_0 p_0 q_0}} , \quad \text{---}\diagup\diagdown\text{---} \quad (\mathcal{S}_1)^{\mathbf{pqk}} = \frac{-2\pi i \ell \delta(\mathbf{p} + \mathbf{q} + \mathbf{k})}{\sqrt{8 l^9 k_0 p_0 q_0}} .$$

Propagators arise when one considers second-order matrix elements, representing processes described, for example, by the diagrams



Here one has two types of second order processes whose initial and final states contain two massive particles. These types are labelled as (I) and (II), and each of them comes in two subtypes, distinguished by the time order of the two interactions involved and respectively labelled as (I') and (I''), (II') and (II''). By considering the form of the interaction, one sees that the first diagram yields a contribution

$$(\mathcal{S}_{I'})_{\mathbf{pq}}^{\mathbf{p'q'}} = \frac{-\ell^2}{l^6} \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} dt_1 \int d^3\mathbf{k} \, \mathcal{H}(t_2 - t_1) \frac{\delta(\mathbf{p}'_{\perp} + \mathbf{q}'_{\perp} - \mathbf{k}_{\perp}) \delta(k_{\perp} - \mathbf{p}_{\perp} - \mathbf{q}_{\perp})}{\sqrt{16 p'_0 q'_0 p_0 q_0} 2 k_0} .$$

$$. e^{i(-p_0 + k_0 - q_0)t_1} e^{i(p'_0 - k_0 + q'_0)t_2} ,$$

where \mathcal{H} is the Heaviside function, arising from the explicit expression of the time-ordered product (which also yields two identical terms, so the $1/2!$ factor in the \mathcal{S} series cancels out).

Now one proceeds essentially in a more or less standard way. First one uses a technical result: if φ is a test function on any fibre of $\mathbf{P}_m \rightarrow \mathbf{T}$, then

$$\int d^3\mathbf{k}_{\perp} \, \mathcal{H}(t) e^{\pm i t E_m(\mathbf{k}_{\perp})} \varphi(\mathbf{k}_{\perp}) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int d^4\mathbf{k} \frac{e^{i t k_0}}{k_0 \mp E_m(\mathbf{k}_{\perp}) - i \varepsilon} \varphi(\mathbf{k}_{\perp}) =$$

$$= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int d^4\mathbf{k} \frac{e^{-i t k_0}}{-k_0 \mp E_m(\mathbf{k}_{\perp}) - i \varepsilon} \varphi(\mathbf{k}_{\perp})$$

(the proof uses the integral representation $2\pi i \mathcal{H}(t) = \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{e^{i t \tau}}{\tau - i \varepsilon} d\tau$ and some integration variable changes). Eventually

$$(\mathcal{S}_{I'})_{\mathbf{pq}}^{\mathbf{p'q'}} = \frac{2\pi i \ell^2}{l^6 \sqrt{16 p'_0 q'_0 p_0 q_0}} \int d^4\mathbf{k} \frac{\delta(\mathbf{p}' + \mathbf{q}' - \mathbf{k}) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q})}{2 |\mathbf{k}_{\perp}| (-k_0 + |\mathbf{k}_{\perp}| - i \varepsilon)}$$

(the limit for $\varepsilon \rightarrow 0^+$ is intended). So, by a technical trick, an integral over $\mathbf{T} \times (\mathbf{P}_m)_{t_0}$, $t_0 \in \mathbf{T}$, was transformed into an integral over a whole space $\mathbb{T}_0^* \mathbf{M}$ of 4-momenta (the momentum of the intermediate particle is ‘off-shell’).

The calculation relative to the diagram I'' is similar but one has a few sign differences, getting

$$(\mathcal{S}_{I''})_{\text{pq}}^{\text{p'q'}} = \frac{2\pi i \ell^2}{l^6 \sqrt{16 \text{p}'_0 \text{q}'_0 \text{p}_0 \text{q}_0}} \int d^4 \mathbf{k} \frac{\delta(\text{p}' + \text{q}' - \mathbf{k}) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q})}{2 |\mathbf{k}_\perp| (k_0 + |\mathbf{k}_\perp| - i\varepsilon)}.$$

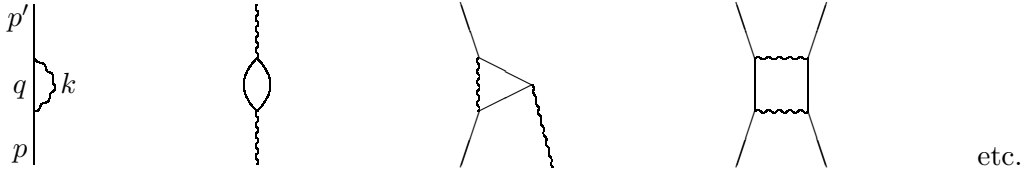
Finally,

$$(\mathcal{S}_I)_{\text{pq}}^{\text{p'q'}} = (\mathcal{S}_I)_{\text{pq}}^{\text{p'q'}} + (\mathcal{S}_{I''})_{\text{pq}}^{\text{p'q'}} = \frac{-2\pi i \ell^2}{l^6 \sqrt{16 \text{p}'_0 \text{q}'_0 \text{p}_0 \text{q}_0}} \int d^4 \mathbf{k} \frac{\delta(\text{p}' + \text{q}' - \mathbf{k}) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q})}{g(\mathbf{k}, \mathbf{k}) + i\varepsilon}.$$

In case II one finds exactly the same result.

When the intermediate particle is massive one gets a similar expression, with $g(\mathbf{k}, \mathbf{k})$ in the propagator’s expression replaced by $g(\mathbf{p}, \mathbf{p}) - m^2$.

A word is due about the ‘infinities’ arising when one considers diagrams containing loops, as for example



Doing the calculation in the first instance (say) one has a contribution to the scattering matrix which, apart from constant factors, is given by the integral

$$\int d^4 \mathbf{k} \frac{\delta(-\mathbf{p} + \mathbf{q} + \mathbf{k}) \delta(-\mathbf{q} - \mathbf{k} + \mathbf{p}')}{4 E_m(\mathbf{q}_\perp) |\mathbf{k}_\perp| (-k_0 + |\mathbf{k}_\perp| - i\varepsilon)}.$$

Now if this were a well-defined distribution ϕ in two variables, then $\langle u \phi v \rangle$ should be a (finite) number, but it is immediate to check that it is not. Similar results are found in the other cases.

7 Electron and positron free states

The 4-spinor bundle is a complex vector bundle $\mathbf{W} \rightarrow \mathbf{M}$ with 4-dimensional fibres, endowed with a *scaled Clifford morphism* (*Dirac map*)

$$\gamma : \mathbf{T}\mathbf{M} \rightarrow \mathbb{L} \otimes \text{End}(\mathbf{W}) : v \mapsto \gamma[v]$$

over \mathbf{M} and a Hermitian metric k on the fibres fulfilling

$$k(\gamma[v]\phi, \psi) = k(\phi, \gamma[v]\psi), \quad v \in \mathbf{T}_x \mathbf{M}, \quad \phi, \psi \in \mathbf{W}_x, \quad x \in \mathbf{M}.$$

Then k (which yields the *Dirac adjoint* anti-isomorphism $\psi \mapsto k^b(\psi)$, usually denoted as $\psi \mapsto \bar{\psi}$) turns out to have the signature $(+, +, -, -)$. If $p : \mathbf{M} \rightarrow \mathbf{P}_m$ then

$$\mathbf{W} = \mathbf{W}_p^+ \oplus_M \mathbf{W}_p^- , \quad \mathbf{W}_p^\pm := \text{Ker}(\gamma[p^\#] \mp m) ,$$

where $p^\# \equiv g^\#(p) : \mathbf{M} \rightarrow \mathbb{L}^{-2} \otimes \mathbf{T}\mathbf{M}$ is the contravariant form of p . The restrictions of k to these two subbundles turn out to have the signatures $(+, +)$ and $(-, -)$, respectively.

Now for each $m \in \{0\} \cup \mathbb{L}^{-1}$ one is led to consider the 2-fibred bundles $\mathbf{W}_m^\pm \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$ defined by

$$\mathbf{W}_m^\pm := \bigsqcup_{p \in \mathbf{P}_m} \mathbf{W}_p^\pm \subset \mathbf{P}_m \times_M \mathbf{W} .$$

The 4-spinor bundle is also endowed⁶ with a *spinor connection* \mathbb{F} , strictly related to the spacetime connection, such that γ and k are covariantly constant. It is easy to see⁷ that Γ_m and \mathbb{F} determine projectable connections of $\mathbf{W}_m^\pm \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$.

Now if $\mathbf{T} \subset \mathbf{M}$ is a detector, then a free one-electron state is defined to be a covariantly constant section $\mathbf{T} \rightarrow \mathbf{W}_m^+$. Namely, a free one-electron state is determined by a covariantly constant section $p : \mathbf{T} \rightarrow \mathbf{P}_m$ and by a covariantly constant section $\psi : \mathbf{T} \rightarrow \mathbf{W}$ such that $\psi(t) \in \mathbf{W}_{p(t)}^+$ for each $t \in \mathbf{T}$. On the other hand, a free one-positron state will be represented as a covariantly constant section $\mathbf{T} \rightarrow \overline{\mathbf{W}}_m^-$.⁸ For brevity, these 2-fibred bundles and the related 1-particle state quantum bundles (of vector-valued generalized half-densities) are denoted as

$$\begin{aligned} \mathbf{F} &\equiv \mathbf{W}_m^+ \rightarrow \mathbf{P}_m \rightarrow \mathbf{T} , & \tilde{\mathbf{F}} &\equiv \overline{\mathbf{W}}_m^- \rightarrow \mathbf{P}_m \rightarrow \mathbf{T} , \\ \mathcal{F}^1 &:= \mathcal{D}_T(\mathbf{P}_m, \mathbf{F}) \equiv \mathcal{D}_T(\mathbf{P}_m, \mathbf{W}_m^+) , & \tilde{\mathcal{F}}^1 &:= \mathcal{D}_T(\mathbf{P}_m, \tilde{\mathbf{F}}) \equiv \mathcal{D}_T(\mathbf{P}_m, \overline{\mathbf{W}}_m^-) . \end{aligned}$$

In order to introduce appropriate generalized frames for free electron and positron states, one needs, for each $p \in (\mathbf{P}_m)_T$ a frame

$$(\mathbf{u}_A(p), \mathbf{v}_A(p)) , \quad A = 1, 2$$

of \mathbf{W}_p which is adapted to the splitting $\mathbf{W}_p = \mathbf{W}_p^+ \oplus \mathbf{W}_p^-$. A consistent choice can be made by extending usual procedure of the flat inertial case.⁹ Now one gets the

⁶See [CJ97a, C00b] for a review of the geometry 4-spinors and 2-spinors for electrodynamics and other field theories.

⁷If $p : \mathbf{M} \rightarrow \mathbf{P}_m$ and $\psi : \mathbf{M} \rightarrow \mathbf{W}$ are parallelly transported along some curve in \mathbf{M} , then $\gamma[p^\#]\psi \mp m\psi$ is also parallelly transported along the same curve; so it vanishes along the curve if it vanishes at any one point of the curve.

⁸If \mathbf{V} is a finite-dimensional complex vector space then its *conjugate space* can be defined as $\overline{\mathbf{V}} := \mathbf{V}^{\star\star} \cong \mathbf{V}^{\star\star}$, where \mathbf{V}^\star and $\mathbf{V}^{\bar{\star}}$ are, respectively, the \mathbb{C} -dual and antidual spaces, that is the spaces of linear and antilinear maps $\mathbf{V} \rightarrow \mathbb{C}$. There is an anti-isomorphism $\mathbf{V} \rightarrow \overline{\mathbf{V}} : v \mapsto \bar{v}$. The indices relative to a conjugate basis are distinguished by a dot.

⁹At some point in \mathbf{T} one fixes a spinor frame adapted to the splitting determined by the unit vector τ_0 , and Fermi transports it along \mathbf{T} ; then, in each fibre, one takes the unique boost sending τ_0 to $p^\# / m$; up to sign (which can be fixed by continuity) this boost transforms the given spinor frame to the desired one.

generalized frames

$$A_{pA} := A_p \otimes u_A(p) : T \rightarrow \mathcal{F}^1 \equiv \mathcal{P}_T(P_m, W_m^+) ,$$

$$C_{pA} := A_p \otimes \bar{v}_A(p) : T \rightarrow \tilde{\mathcal{F}}^1 \equiv \mathcal{P}_T(P_m, \overline{W}_m^-) ,$$

respectively for electrons and positrons.

An important technical result, which is proved by elementary linear algebra, is

$$u_A(p) \otimes u^A(p) = \frac{1}{2} (\mathbb{1} + \gamma[p^\# / m]) : W \rightarrow W_p^+ ,$$

$$v_A(p) \otimes v^A(p) = \frac{1}{2} (\mathbb{1} - \gamma[p^\# / m]) : W \rightarrow W_p^- .$$

8 Photon free states

For brevity, henceforth I will use the shorthand $\mathbf{H} \equiv \mathbb{L}^{-1} \otimes T\mathbf{M}$, so that $\mathbf{H}^* \equiv \mathbb{L} \otimes T^*\mathbf{M}$ and the spacetime metric g is an *unscaled* (i.e. ‘conformally invariant’) Lorentz metric in the fibres of $\mathbf{H} \rightarrow \mathbf{M}$.

Remember that $P_0 \subset T^*\mathbf{M}$ denotes the subbundle over \mathbf{M} of future null half-cones in the fibres of $T^*\mathbf{M}$. Consider the 2-fibred bundle $\mathbf{H}_{0\perp} \rightarrow P_0 \rightarrow \mathbf{M}$ whose fibre over any $k \in (P_0)_x$, $x \in \mathbf{M}$, is the 3-dimensional real vector space

$$(\mathbf{H}_{0\perp})_k := \{\alpha \in \mathbf{H}^* : g^\#(\alpha, k) = 0\} .$$

Then $\mathbf{H}_{0\perp} \subset P_0 \times_M \mathbf{H}^*$ (but note that $\mathbf{H}_{0\perp}$ itself is *not* a ‘semi-trivial’ bundle of the type $P_0 \times_M \mathbf{Z}$). Next, consider the real vector bundle $\mathbf{B}_\mathbb{R}^* \rightarrow P_0$ whose fibre over any $k \in P_0$ is the (2-dimensional) quotient space

$$(\mathbf{B}_\mathbb{R})_k^* := (\mathbf{H}_{0\perp})_k / \langle k \rangle \equiv \langle k \rangle^\perp / \langle k \rangle ,$$

where $\langle k \rangle$ denotes the vector space generated by k . Moreover, $\mathbf{B}_\mathbb{R} \equiv \mathbf{B}_\mathbb{R}^{**}$ can be equivalently introduced by a similar contravariant construction.

It turns out that the spacetime metric ‘passes to the quotient’, so it naturally determines a negative metric g_B in the fibres of $\mathbf{B}_\mathbb{R} \rightarrow P_0$, as well as a ‘Hodge’ isomorphism $*_B$ which can be characterized through the rule¹⁰

$$*(k \wedge \beta) = -k \wedge (*_B \beta) .$$

Now define the *optical bundle*¹¹ to be the 2-fibred bundle

$$\mathbf{B} := \mathbb{C} \otimes \mathbf{B}_\mathbb{R} \rightarrow P_0 \rightarrow \mathbf{M} .$$

¹⁰ $k \wedge \beta$ is well defined because β is an equivalence class of covectors differing for a term proportional to k .

¹¹In the literature this term is often used in a somewhat different (but related) sense, denoting a vector bundle over \mathbf{M} associated with the choice of a congruence of null lines [Nu96].

This has the canonical splitting

$$\mathbf{B} = \mathbf{B}^+ \oplus_{P_0} \mathbf{B}^- ,$$

where the fibres of the subbundles $\mathbf{B}^\pm \rightarrow P_0$ are defined to be the eigenspaces of $-\mathbf{i} *_{\mathbf{B}}$ with eigenvalues ± 1 (and turn out to be 1-dimensional complex $g_{\mathbf{B}}$ -null subspaces).

Let $u : M \rightarrow TM$ be any given ‘observer’, i.e. a unit timelike vector field on M ; through u one can identify $\mathbf{B}_{\mathbb{R}} \rightarrow P_0$ with $\mathbf{H}_{0\perp} \cap \langle u \rangle^\perp \rightarrow P_0$ (‘radiation gauge’). Take any $k \in (P_0)_x$, $x \in M$, and let (\mathbf{e}_λ) , $\lambda = 0, 1, 2, 3$, be an orthonormal basis of \mathbf{H}_x such that $\mathbf{e}_0 \equiv u(x)$ and $k^\# \propto \mathbf{e}_0 + \mathbf{e}_3$; then the basis

$$(\mathbf{b}_1, \mathbf{b}_2) \equiv (\mathbf{b}_+, \mathbf{b}_-) := \left(\frac{1}{\sqrt{2}} (\mathbf{e}_1 + \mathbf{i} \mathbf{e}_2) , \frac{1}{\sqrt{2}} (\mathbf{e}_1 - \mathbf{i} \mathbf{e}_2) \right) \subset \mathbb{C} \otimes (\mathbf{H}_{0\perp} \cap \langle u \rangle^\perp)$$

is *adapted* to the splitting of \mathbf{B}_k , i.e. $\mathbf{b}_\pm \in \mathbf{B}_k^\pm$. In this way, locally one can construct smooth frames of $\mathbb{C} \otimes \mathbf{H} \times_M P_0$, and smooth frames of $\mathbf{B} \rightarrow P_0$ which are adapted to its splitting.

Let now $\{\mathbf{B}_k\}$ be the generalized frame of the quantum bundle $\mathcal{P}_0 \rightarrow M$ defined as usual; then one gets a generalized frame

$$\{\mathbf{B}_{\kappa Q}\} := \{\mathbf{B}_k \otimes \mathbf{b}_Q(k)\} , \quad Q = 1, 2 ,$$

of the quantum bundle

$$\mathcal{B}^1 := \mathcal{P}_M(P_0, \mathbf{B}) \rightarrow M .$$

In particular, all the above bundles and constructions can be restricted to a detector $T \subset M$. In that case, \mathbf{e}_0 will be chosen to be the unit future-pointing vector tangent to T .

Free asymptotic 1-photon states will be described as covariantly constant sections $T \rightarrow \mathcal{B}^1$ (they only possess *transversal polarization modes*). Virtual photons, on the other hand, span a larger bundle; they are described as covariantly constant sections

$$T \rightarrow \tilde{\mathcal{B}}^1 \equiv \mathcal{P}_T(P_0, \mathbb{C} \otimes \mathbf{H}) ,$$

where one uses the generalized frame

$$\{\mathbf{B}_{\kappa\lambda}\} := \{\mathbf{B}_k \otimes \mathbf{e}_\lambda\} , \quad \lambda = 0, 1, 2, 3 .$$

9 Electromagnetic interaction

The ‘classical electromagnetic interaction’ is the 3-linear morphism

$$\begin{aligned} \ell_{\text{int}} : \overline{\mathbf{W}} \times_M \mathbf{H} \times_M \mathbf{W} &\rightarrow \mathbb{C} \\ &: (\phi, b, \psi) \mapsto -e k(\phi, \gamma[b]\psi) \equiv -e \langle k^\flat \phi, \gamma[b]\psi \rangle , \end{aligned}$$

where $e \in \mathbb{R}^+$ is the positron's charge (a pure number in natural units).

As sketched in §5, this geometric structure of the underlying classical bundles, together with the generalized half-density $\underline{\Delta}$, determines the quantum interaction $-i\mathfrak{H}$. A short discussion is needed in order to see how the index types of the various terms in \mathfrak{H} arise.

First, ℓ_{int} is extended to $\overline{\mathbf{W}} \times_M \mathbf{H}_{\mathbb{C}} \times_M \mathbf{W}$, where $\mathbf{H}_{\mathbb{C}} \equiv \mathbb{C} \otimes \mathbf{H}$. In the fibres of the complex vector bundle $\mathbf{H}_{\mathbb{C}} \rightarrow \mathbf{M}$ indices are raised and lowered through the obvious extension of the spacetime metric g , while in the fibres of $\mathbf{B} \rightarrow \mathbf{P}_0$ one uses g_B ; when an observer is chosen and one works in the radiation gauge, the latter operation can be viewed essentially as a restriction of the former.

Now ℓ_{int} can be seen as a \mathbb{C} -linear function on the fibres of

$$\overline{\mathbf{W}} \otimes_{P_{\Delta}} \mathbf{H}_{\mathbb{C}} \otimes_{P_{\Delta}} \mathbf{W} \rightarrow P_{\Delta} \equiv P_m \times_M P_0 \times_M P_m .$$

Note that the isomorphism $k^b : \overline{\mathbf{W}} \rightarrow \mathbf{W}^{\star}$ induced by the Hermitian metric k preserves the splitting $\mathbf{W} \times_M P_m = \mathbf{W}_m^+ \oplus_{P_m} \mathbf{W}_m^-$, namely $k^b : \overline{\mathbf{W}}_m^{\pm} \rightarrow (\mathbf{W}_m^{\pm})^{\star}$. Then

$$P_m \times_M \overline{\mathbf{W}} = \overline{\mathbf{W}}_m^+ \oplus_{P_m} \overline{\mathbf{W}}_m^- \cong (\mathbf{W}_m^+)^{\star} \oplus_{P_m} \overline{\mathbf{W}}_m^- ,$$

$$P_m \times_M \mathbf{W} = \mathbf{W}_m^+ \oplus_{P_m} \mathbf{W}_m^- \cong \mathbf{W}_m^+ \oplus_{P_m} (\overline{\mathbf{W}}_m^-)^{\star} .$$

When $P_m \times_M \mathbf{W}$ and $P_m \times_M \overline{\mathbf{W}}$ are written in this way, the coordinate expression of

$$\ell_{\text{int}} : \left((\mathbf{W}_m^+)^{\star} \oplus_{P_m} \overline{\mathbf{W}}_m^- \right) \otimes_{P_{\Delta}} \mathbf{H}_{\mathbb{C}} \otimes_{P_{\Delta}} \left(\mathbf{W}_m^+ \oplus_{P_m} (\overline{\mathbf{W}}_m^-)^{\star} \right) \rightarrow \mathbb{C}$$

contains four terms with different index types; all dotted (i.e. ‘conjugated’) indices, either high or low, refer to the positron bundle $\overline{\mathbf{W}}_m^-$ or to its dual, while undotted indices refer to the electron bundle or to its dual. Finally, a further extension through g gives

$$\ell_{\text{int}} : \left((\mathbf{W}_m^+)^{\star} \oplus_{P_m} \overline{\mathbf{W}}_m^- \right) \otimes_{P_{\Delta}} \left(\mathbf{H}_{\mathbb{C}} \oplus_{P_0} \mathbf{H}_{\mathbb{C}}^{\star} \right) \otimes_{P_{\Delta}} \left(\mathbf{W}_m^+ \oplus_{P_m} (\overline{\mathbf{W}}_m^-)^{\star} \right) \rightarrow \mathbb{C} ,$$

which is the sum of *eight* terms of different index types. Explicitely, if $\beta \in \mathbf{H}_{\mathbb{C}}$ then one replaces $\gamma[b]$ in the expression of ℓ_{int} with

$$\gamma^{\#}[\beta] := \gamma[\beta^{\#}] \equiv \gamma[g^{\#}\beta] .$$

Further objects can be obtained by exchanging tensor factors in ℓ_{int} . However, objects only distinguished for a different order of indices referring to different particle types are regarded as equivalent, while in the different ordering of indices referring to the same particle type only those terms are retained which have the covariant indices *on the right* of the contravariant ones.

Let now $\mathbf{T} \subset \mathbf{M}$ be a detector, consider the restrictions to \mathbf{T} of the various quantum bundles and the Fock bundle $\tilde{\mathcal{F}} \otimes_{\mathbf{T}} \tilde{\mathcal{B}} \otimes_{\mathbf{T}} \mathcal{F}$. At this point, one has all the ingredients needed to write down the interaction morphism over \mathbf{T}

$$\mathfrak{H} : \tilde{\mathcal{F}}_{\circ} \otimes_{\mathbf{T}} \tilde{\mathcal{B}}_{\circ} \otimes_{\mathbf{T}} \mathcal{F}_{\circ} \rightarrow \mathbb{L}^{-1} \otimes \tilde{\mathcal{F}} \otimes_{\mathbf{T}} \tilde{\mathcal{B}} \otimes_{\mathbf{T}} \mathcal{F} ,$$

where the subscript circles indicate the subbundles of test elements. The constant $\lambda \in \mathbb{L}^{-1}$ introduced at the end of §5 is, here, the electron's mass m . Thus one finds

$$\begin{aligned} \frac{1}{m} \mathfrak{H} = & e^{i(-p_0-k_0-q_0)t} \Lambda_{pA'k\lambda qB} C^{pA'} \otimes B^{k\lambda} \otimes A^{qB} + & \text{Diagram 1} \\ & + e^{i(p_0-k_0-q_0)t} \Lambda_{k\lambda qB}^{pA} A_{pA} \otimes B^{k\lambda} \otimes A^{qB} + & \text{Diagram 2} \\ & + e^{i(-p_0+k_0-q_0)t} \Lambda_{pA' qB}^{k\lambda} C^{pA'} \otimes B_{k\lambda} \otimes A^{qB} + & \text{Diagram 3} \\ & + e^{i(p_0-k_0-q_0)t} \Lambda_{k\lambda qB'}^{pA'} C_{pA'} \otimes B^{k\lambda} \otimes C^{qB'} + & \text{Diagram 4} \\ & + e^{i(p_0+k_0-q_0)t} \Lambda_{qB}^{pA k\lambda} A_{pA} \otimes B_{k\lambda} \otimes A^{qB} + & \text{Diagram 5} \\ & + e^{i(p_0-k_0+q_0)t} \Lambda_{k\lambda}^{pA qB'} A_{pA} \otimes B^{k\lambda} \otimes C_{qB'} + & \text{Diagram 6} \\ & + e^{i(p_0+k_0-q_0)t} \Lambda_{qB'}^{pA' k\lambda} C_{pA'} \otimes B_{k\lambda} \otimes C^{qB'} + & \text{Diagram 7} \\ & + e^{i(p_0+k_0+q_0)t} \Lambda_{qB'}^{pA k\lambda qB'} A_{pA} \otimes B_{k\lambda} \otimes C_{qB'} . & \text{Diagram 8} \end{aligned}$$

where

$$\Lambda_{pA'k\lambda qB} = \ell_{pA'k\lambda qB} \underline{\Lambda}_{p k q} = \ell_{pA'k\lambda qB} \frac{\delta(p_{\perp} + k_{\perp} + q_{\perp})}{\sqrt{8 l^9 p_0 k_0 q_0}}$$

and the like. Explicitely, the ℓ -factors are given by

$$\begin{aligned} \ell_{pA'k\lambda qB} &= -e \bar{v}_{pA'} \gamma_{k\lambda} u_{qB} , & \ell_{k\lambda qB}^{pA} &= -e u^{pA} \gamma_{k\lambda} u_{qB} , \\ \ell_{pA' qB}^{k\lambda} &= -e \bar{v}_{pA'} \gamma^{k\lambda} u_{qB} , & \ell_{k\lambda qB'}^{pA'} &= -e \bar{v}_{qB'} \gamma_{k\lambda} \bar{v}^{pA'} , \\ \ell_{qB}^{pA k\lambda} &= -e u^{pA} \gamma^{k\lambda} u_{qB} , & \ell_{k\lambda}^{pA qB'} &= -e u^{pA} \gamma_{k\lambda} \bar{v}^{qB'} , \\ \ell_{qB'}^{pA' k\lambda} &= -e \bar{v}_{qB'} \gamma^{k\lambda} \bar{v}^{pA'} , & \ell_{qB'}^{pA k\lambda qB'} &= -e u^{pA} \gamma^{k\lambda} \bar{v}^{qB'} , \end{aligned}$$

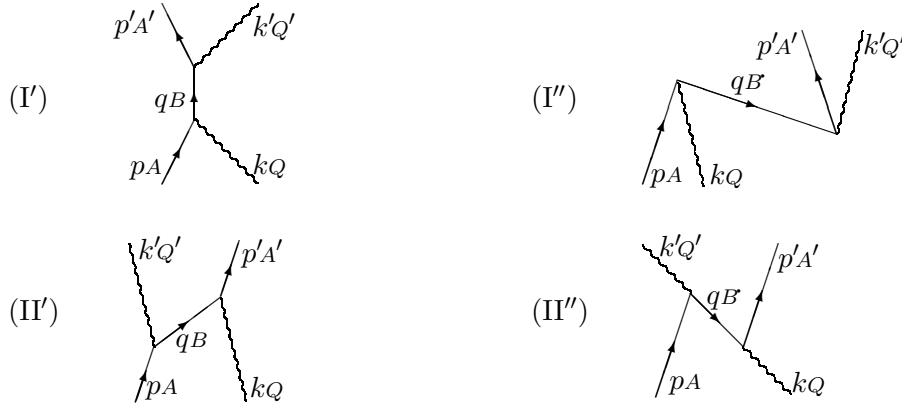
where $e_{k\lambda} \equiv e_{\lambda}(k)$, $e^{k\lambda} \equiv e^{\lambda}(k)$ denotes its dual frame of $\mathbf{H}_{\mathcal{C}}^{\star}$. Moreover, $\gamma_{k\lambda} \equiv \gamma[e_{k\lambda}]$ and $\gamma^{k\lambda} \equiv \gamma^{\#}[e^{k\lambda}]$.

In the above elementary diagrams time runs upwards; so, lines entering the vertex from below represent absorbed particles, lines entering from above represent created particles; electron lines are labelled by up arrows, positron lines are labelled by down arrows, and photon lines are wavy.

10 QED

In this section I will show how two-point interactions¹² give rise to scattering matrix contributions which, at least formally, have the same expressions as in standard treatments; these expressions are the so-called *propagators* of the particles in momentum space. In the flat inertial case one recovers standard results.

Consider a second order process in which the initial and final states both contain one electron and one photon. One has two types of diagrams, and for each type on turn two subtypes can be distinguished, according to the time order of the vertices:



Here, external photon lines are labelled by an index $Q = 1, 2$ referring to the classical frame (b_{k_Q}) of the bundle $\mathbf{B} \rightarrow \mathbf{P}_0$ of transversal polarization modes; 4-momenta are indicated by letters p, q, k etcetera.

First, consider the diagram labelled as (I'). Following the usual procedure one finds

$$\begin{aligned}
 (\mathcal{S}_{I'})^{p'A'k'Q'}_{p_A k_Q} &= -m^2 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} dt_1 \int d^3 \mathbf{q}_\perp H(t_2 - t_1) \left(\sum_{B=1}^2 \ell^{p'A'}_{q_B} \ell^{k'Q'}_{p_A k_Q} \right) \cdot \\
 &\quad \cdot \frac{\delta(\mathbf{p}'_\perp - \mathbf{q}_\perp + \mathbf{k}'_\perp) \delta(-\mathbf{p}_\perp + \mathbf{q}_\perp - \mathbf{k}_\perp)}{l^6 \sqrt{16 p'_0 k'_0 p_0 k_0} 2 q_0} \cdot e^{i(-p_0 + q_0 - k_0)t_1} e^{i(p'_0 - q_0 + k'_0)t_2} .
 \end{aligned}$$

Note that the summation over B , here, is to be performed before all other operations: it must be performed before integration over \mathbf{q} because the index B ‘resides’ over \mathbf{q} ,

¹²One-point interactions in QED are nearly obvious at this stage.

and before the transformation of the integral into a 4-dimensional one because the index B cannot reside over an off-shell momentum. Hence one considers, for *fixed* \mathbf{q} ,

$$\begin{aligned} \sum_{B=1}^2 \ell^{\mathbf{p}'A'}_{\mathbf{q}B} \ell^{\mathbf{k}'Q'}_{\mathbf{p}A} \ell^{\mathbf{q}B}_{\mathbf{k}Q} &= e^2 \sum_{B=1}^2 (u^{\mathbf{p}'A'} \gamma^{\mathbf{k}'Q'} u_{\mathbf{q}B}) (u^{\mathbf{q}B} \gamma_{\mathbf{k}Q} u_{\mathbf{p}A}) = \\ &= e^2 \sum_{B=1}^2 u^{\mathbf{p}'A'} \circ \gamma^{\mathbf{k}'Q'} \circ (u_{\mathbf{q}B} \otimes u^{\mathbf{q}B}) \circ \gamma_{\mathbf{k}Q} u_{\mathbf{p}A} = \\ &= e^2 u^{\mathbf{p}'A'} \circ \gamma^{\mathbf{k}'Q'} \circ (\mathbb{1} + \frac{1}{m} \gamma^\#[\mathbf{q}]) \circ \gamma[\mathbf{b}_{\mathbf{k}Q}] u_{\mathbf{p}A} , \end{aligned}$$

since $\sum_{B=1}^2 u_{\mathbf{q}B} \otimes u^{\mathbf{q}B}$, for fixed \mathbf{q} , is just the projection $\mathbb{1} + \frac{1}{m} \gamma^\#[\mathbf{q}] : \mathbf{W} \rightarrow \mathbf{W}_{\mathbf{q}}^+$. Furthermore, observe that for $\mathbf{q} \in \mathbf{P}_m$ one has

$$\mathbb{1} + \frac{1}{m} \gamma^\#[\mathbf{q}] = \frac{1}{m} (m + E_m(\mathbf{q}_\perp) \gamma^0 + \gamma^\#[\mathbf{q}_\perp]) , \quad E_m(q_\perp) := \sqrt{m^2 + |\mathbf{q}_\perp|^2} .$$

Now when one performs the usual trick for transforming the integral into a 4-dimensional one, the above factor remains unchanged, so that

$$\begin{aligned} (\mathcal{S}_{I'})^{\mathbf{p}'A' \mathbf{k}'Q'}_{\mathbf{p}A \mathbf{k}Q} &= \frac{2\pi i m e^2}{l^6 \sqrt{16 \mathbf{p}'_0 \mathbf{k}'_0 \mathbf{p}_0 \mathbf{k}_0}} u^{\mathbf{p}'A'} \circ \gamma^{\mathbf{k}'Q'} \circ \\ &\circ \left(\int d^4 \mathbf{q} \frac{\delta(-\mathbf{p} - \mathbf{k} + \mathbf{q}) \delta(-\mathbf{q} + \mathbf{p}' + \mathbf{k}')}{2 E_m(\mathbf{q}_\perp) (-\mathbf{q}_0 + E_m(\mathbf{q}_\perp) - i\varepsilon)} (m + E_m(\mathbf{q}_\perp) \gamma^0 + \gamma^\#[\mathbf{q}_\perp]) \right) \circ \gamma_{\mathbf{k}Q} u_{\mathbf{p}A} . \end{aligned}$$

Next, consider the diagram labelled as (I''). Like in the scalar case, the different time order of the vertices yields different signs in the arguments of the Dirac deltas. The classical Lagrangian yields now a further difference, since

$$\sum_{B'=1}^2 \ell^{\mathbf{p}'A' \mathbf{k}'Q'}_{\mathbf{q}B'} \ell_{\mathbf{p}A \mathbf{k}Q \mathbf{q}B'} = e^2 u^{\mathbf{p}'A'} \circ \gamma^{\mathbf{k}'Q'} \circ (\mathbb{1} - \frac{1}{m} \gamma^\#[\mathbf{q}]) \circ \gamma_{\mathbf{k}Q} u_{\mathbf{p}A} .$$

Then one finds

$$\begin{aligned} (\mathcal{S}_{I''})^{\mathbf{p}'A' \mathbf{k}'Q'}_{\mathbf{p}A \mathbf{k}Q} &= \frac{2\pi i m e^2}{l^6 \sqrt{16 \mathbf{p}'_0 \mathbf{k}'_0 \mathbf{p}_0 \mathbf{k}_0}} u^{\mathbf{p}'A'} \circ \gamma^{\mathbf{k}'Q'} \circ \\ &\circ \left(\int d^4 \mathbf{q} \frac{\delta(-\mathbf{p}_\perp - \mathbf{k}_\perp - \mathbf{q}_\perp) \delta(\mathbf{q}_\perp + \mathbf{p}'_\perp + \mathbf{k}'_\perp)}{2 E_m(\mathbf{q}_\perp) (\mathbf{q}_0 + E_m(\mathbf{q}_\perp) - i\varepsilon)} \delta(-\mathbf{p}_0 - \mathbf{k}_0 + \mathbf{q}_0) \delta(-\mathbf{q}_0 + \mathbf{p}'_0 + \mathbf{k}'_0) \cdot \right. \\ &\quad \left. \cdot (m - E_m(\mathbf{q}_\perp) \gamma^0 - \gamma^\#[\mathbf{q}_\perp]) \right) \circ \gamma_{\mathbf{k}Q} u_{\mathbf{p}A} . \end{aligned}$$

In order to simplify $(\mathcal{S}_I)^{\mathbf{p}'A' \mathbf{k}'Q'}_{\mathbf{p}A \mathbf{k}Q} = (\mathcal{S}_{I'} + \mathcal{S}_{I''})^{\mathbf{p}'A' \mathbf{k}'Q'}_{\mathbf{p}A \mathbf{k}Q}$ one has to make the integration variable change $\mathbf{q}_\perp \rightarrow -\mathbf{q}_\perp$ in the second contribution, so that the δ -

factors are the same. Eventually,

$$(\mathcal{S}_I)^{p'A'k'Q'}_{pAkQ} = \frac{-2\pi i m e^2}{l^6 \sqrt{16 p'_0 k'_0 p_0 k_0}} u^{p'A'} \circ \gamma^{k'Q'} \circ$$

$$\circ \left(\int d^4 q \delta(-p - k + q) \delta(-q + p' + k') \cdot \frac{m + \gamma^\# [q]}{g(q, q) - m^2 + i\varepsilon} \right) \circ \gamma_{kQ} u_{pA} ,$$

which contains the *electron propagator*, namely the distribution

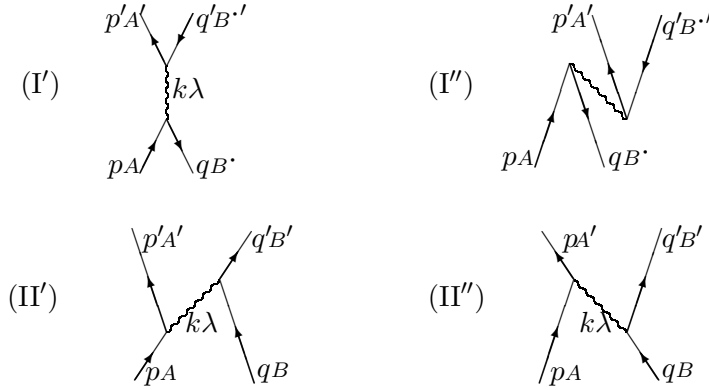
$$\lim_{\varepsilon \rightarrow 0^+} \frac{-m - \gamma^\# [q]}{g(q, q) - m^2 + i\varepsilon} .$$

The *positron propagator*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{-m + \gamma^\# [q]}{g(q, q) - m^2 + i\varepsilon} .$$

is found by a similar procedure.

Next, consider the diagrams



From case (II) one can obtain two further similar cases by inverting one or both fermion paths. In all cases the calculation is essentially the same; case (II) is somewhat simpler notationally since it has no dotted indices. Diagram (II') yields the summation

$$\sum_{\lambda=0}^3 \ell^{p'A'k\lambda}_{pA} \ell^{q'B'k\lambda}_{k\lambda qB} = e^2 \sum_{\lambda=0}^3 (u^{p'A'} \gamma^{k\lambda} u_{pA}) (u^{q'B'} \gamma_{k\lambda} u_{qB})$$

over the internal polarization degrees of freedom of the photon; the generalized index k is kept fixed (no summation on it). Here $\gamma_{k\lambda} \equiv \gamma[e_{k\lambda}]$ and $\gamma^{k\lambda} \equiv \gamma^\# [e^{k\lambda}]$.

In order to handle the above expression conveniently, look at the Dirac map γ as a linear morphism $\mathbf{H} \rightarrow \mathbf{W} \otimes \mathbf{W}^\star$, so that

$$\gamma[y] \otimes \gamma[y] \in (\mathbf{W} \otimes \mathbf{W}^\star) \otimes (\mathbf{W} \otimes \mathbf{W}^\star) = \text{Lin}(\mathbf{W}^\star \otimes \mathbf{W}, \mathbf{W} \otimes \mathbf{W}^\star) , \quad y \in \mathbf{H} .$$

One then finds

$$\gamma^{k\lambda} \otimes \gamma_{k\lambda} = g^{\lambda\mu} \gamma_{k\lambda} \otimes \gamma_{k\mu} = g_{\lambda\mu} \gamma^{k\lambda} \otimes \gamma^{k\mu} : \mathbf{W}^\star \otimes \mathbf{W} \rightarrow \mathbf{W} \otimes \mathbf{W}^\star .$$

Moreover, note that the generalized index k in the above expression *can be dropped*, since the described object is independent of the frame in which it is written, namely

$$\gamma^{k\lambda} \otimes \gamma_{k\lambda} = \gamma^{k'\lambda'} \otimes \gamma_{k'\lambda'} \equiv \gamma^\lambda \otimes \gamma_\lambda .$$

Now the previously considered summation over the virtual photon's polarization states can be rewritten as

$$\sum_{\lambda=0}^3 \ell^{p'A'k\lambda}_{p_A} \ell^{q'B'k\lambda}_{q_B} = e^2 g_{\lambda\mu} (u^{p'A'} \otimes u_{p_A}) \circ (\gamma^\lambda \otimes \gamma^\mu) \circ (u^{q'B'} \otimes u_{q_B}) .$$

From diagram (II') one gets, similarly,

$$\sum_{\lambda=0}^3 \ell^{p'A'k\lambda}_{k\lambda p_A} \ell^{q'B'k\lambda}_{q_B} = e^2 g_{\lambda\mu} (u^{p'A'} \otimes u_{p_A}) \circ (\gamma^\lambda \otimes \gamma^\mu) \circ (u^{q'B'} \otimes u_{q_B}) .$$

Thus, eventually, the contraction over the internal states of the virtual photon gives the same result in the two subcases (II') and (II'') differing for the time ordering of the interaction. The fact that the generalized index k disappears in this operation implies that the photon propagator is simply the scalar massless one tensorialized by the spacetime metric, that is

$$\begin{aligned} (\mathcal{S}_{II})^{p'A'q'B'}_{p_A q_B} &= \frac{-2\pi i m^2 e^2}{l^6 \sqrt{16 p'_0 q'_0 p_0 q_0}} (u^{p'A'} \otimes u_{p_A}) \circ \gamma^\lambda \circ \\ &\circ \left(\int d^4 k \delta(-p - k + q) \delta(-q + p' + k') \cdot \frac{g_{\lambda\mu}}{g(k, k) + i\varepsilon} \right) \cdot \gamma^\mu (u^{q'B'} \otimes u_{q_B}) . \end{aligned}$$

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